MATHM5195 EXERCISE SHEET 5
SOLUTIONS

DUE: MAY 6, 2024

Algebraic geometry, Gröbner bases

Problem 1. (a) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be two algebraic sets, and let

$$X \times Y = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{A}^{n+m} : (x_1, \ldots, x_n) \in X \ (y_1, \ldots, y_m) \in Y\}$$

be their Cartesian product. Show that $X \times Y$ is an algebraic set.

(b) If both X and Y are irreducible, then is $X \times Y$ is irreducible?

Solution. (a) We may assume that $X \subset \mathbb{A}^n$ is $V(f_1, \ldots, f_k)$ where $f_i \in K[x_1, \ldots, x_n]$ for $i = 1, \ldots, k$ and $Y \subset \mathbb{A}^m$ is $V(g_1, \ldots, g_l)$ for $g_i \in K[y_1, \ldots, y_m]$, $i = 1, \ldots, l$.

Note that we can regard the f_i and g_i as elements in $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$, e.g., via defining $\tilde{f}_i(x_1, \ldots, x_n, y_1, \ldots, y_m) := f_i(x_1, \ldots, x_n)$ and $\tilde{g}_i(x_1, \ldots, x_n, y_1, \ldots, y_m) := g_i(y_1, \ldots, y_m)$.

Then $W := V(\tilde{f}_1, \ldots, \tilde{f}_k, \tilde{g}_1, \ldots, \tilde{g}_l) \subseteq \mathbb{A}^{n+m}$ is an algebraic set. In the following write shorthand (x, y) for $(x_1, \ldots, x_n, y_1, \ldots, y_m)$. We have

$$\begin{aligned} W &= \{(x,y) \in \mathbb{A}^{n+m} : \tilde{f}_1(x,y) = \ldots = \tilde{f}_k(x,y) = \tilde{g}_1(x,y) = \ldots = \tilde{g}_l(x,y) = 0\} \\ &= \{(x,y) \in \mathbb{A}^{n+m} : f_1(x) = \ldots = f_k(x) = 0 \text{ and } g_1(y) = \ldots = g_l(y) = 0\} \\ &= \{(x,y) \in \mathbb{A}^{n+m} : x \in X \text{ and } y \in Y\}. \end{aligned}$$

This shows that $W = X \times Y$.

(b) YES. Here is a proof.

We assume that $X \times Y = Z_1 \cup Z_2$ for some algebraic sets Z_i and $Z_i \subsetneq X \times Y$. We show that this implies that X is reducible, a contradition:

First, for $x \in X$ the set $\{x\} \times Y$ is irreducible (it is in fact, isomorphic to Y). We can write

$$\{x\} \times Y = (Z_1 \cap (\{x\} \times Y)) \cup (Z_2 \cap (\{x\} \times Y)).$$

Since $\{x\} \times Y$ is irreducible, it is either contained in Z_1 or in Z_2. Now define $X_i := \{x \in X : \{x\} \times Y \subseteq Z_i\}$ for $i = 1, 2$. Clearly, $X = X_1 \cup X_2$ and $X_i \subsetneq X$, since the Z_i are irreducible. It remains to show that the X_i are closed.

Note that the set $X \times \{y\}$ either lies entirely in Z_1 or in Z_2 for any $y \in Y$ (see this like above, or alternatively, by showing that $X \times \{y\} = Z_i \cap (\mathbb{A}^n \times \{y\})$ for $i = 1$ or $i = 2$). So the set $Z_i \cap (X \times \{y\}) = \{x \in X : (x, y) \subset Z_i\}$ is closed for any $y \in Y$.

Consider the isomorphism $\varphi : X \to X \times \{y\}$. This is a morphism of affine algebraic varieties and one can show that it is continuous (see e.g., Ravi Vakil's lecture notes: `https://math.stanford.edu/~vakil/725/class4.pdf`).

Then $\varphi^{-1}(Z_i \cap (X \times \{y\})) = X_i$ is closed in X (as the preimage of a closed set is closed).

Problem 2. (a) Show (by an example) that an infinite union of algebraic sets is not necessarily an algebraic set.

(b) Give an example of a maximal ideal J in $\mathbb{R}[x_1, \dots, x_n]$ such that $V(J) = \varnothing$. Why does this not contradict the Nullstellensatz?

Solution. (a) Consider $\mathbb{A}^1_{\mathbb{R}}$. Then each $z \in \mathbb{Z}$ is an algebraic subset of $\mathbb{A}^1_{\mathbb{R}}$: $\{z\} = V(x - z)$, where $x - z \in \mathbb{R}[x]$. But $\mathbb{Z} = \bigcup_{z \in \mathbb{Z}} V(x - z)$ is not an algebraic subset of $\mathbb{A}^1_{\mathbb{R}}$, since if there was a polynomial $f \in \mathbb{R}[x]$ vanishing on every integer, it would have $\deg(f) = \infty$. Contradiction.

(b) Let $J = \langle x^2 + 1 \rangle$. Then J is maximal because $\mathbb{R}[x]/J \cong \mathbb{C}$ is a field. But $f(x) = x^2 + 1 > 0$ for any $x \in \mathbb{R}$. This does not contradict the Nullstellensatz because $\mathbb{R}$ is not an algebraically closed field.

Problem 3. (a) Determine the cardinality of $V(f)$ where $f(z) = z^5 - z^4 + z^3 - 1$ is in $\mathbb{C}[z]$ and compare it to $\dim_{\mathbb{C}}(\mathbb{C}[z]/\langle z^5 - z^4 + z^3 - 1 \rangle)$ (dimension here means vector space dimension).

(b) Same question for $V(x - 2y, y^2 - x^3 + x^2 + x)$ and $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/\langle x - 2y, y^2 - x^3 + x^2 + x \rangle$. Geometric interpretation?

(c) Same question for $V(x^3 - yz, y^2 - xz, z^2 - x^2y)$ and $\dim_{\mathbb{C}}(\mathbb{C}[x, y, z]/\langle x^3 - yz, y^2 - xz, z^2 - x^2y \rangle$. (Hint: Recall that $\dim_{\mathbb{C}}(\mathbb{C}[t]) = \infty$ and so also for any $\mathbb{C}$-module containing $\mathbb{C}[t]$)

Solution. (a) Since f is a complex polynomial, it has exactly 5 zeros. A computation (e.g. in Maple) shows that all five zeros are different. On the other hand $\mathbb{C}[z]/\langle z^5 - z^4 + z^3 - 1 \rangle \cong \mathbb{C}z^4 \oplus \mathbb{C}z^3 \oplus \mathbb{C}z^2 \oplus \mathbb{C}z \oplus \mathbb{C}$, so its vector space dimension is also 5.

(b) In order to get $V(x - 2y, y^2 - x^3 + x^2 + x)$, we solve the system of equations $x = 2y$ and $y^2 - x^3 + x^2 + x = 0$. Substituting the first equation into the second one, we see that x is one of the three values: $x_1 = 0$, $x_2 = 5/2 + \frac{\sqrt{29}}{2}$ or $x_3 = 5/2 - \frac{\sqrt{29}}{2}$.

So we get that

$$V(x - 2y, y^2 - x^3 + x^2 + x) = \{(0, 0\} \cup \{(5/2 + \frac{\sqrt{29}}{2}, 5/4 + \frac{\sqrt{29}}{4})\} \cup \{(5/2 - \frac{\sqrt{29}}{2}, 5/4 - \frac{\sqrt{29}}{4})\} .$$

Similarly as above we see that $\mathbb{C}[x, y]/\langle x - 2y, y^2 - x^3 + x^2 + x \rangle \cong \mathbb{C}[x]/\langle x^3 - 5/4x^2 - x \rangle \cong \mathbb{C}x^2 \oplus \mathbb{C}x \oplus \mathbb{C}$. So again the two numbers are equal.

(c) For $V(x^3 - yz, y^2 - xz, z^2 - x^2y)$ we can check that all points of the form (t^3, t^4, t^5) for any $t \in \mathbb{C}$ are contained in this algebraic set.

We can find the surjective ring homomorphism $\varphi : \mathbb{C}[x,y,z] \to \mathbb{C}[t^3,t^4,t^5]$ that sends $x \mapsto t^3, y \mapsto t^4, z \mapsto t^5$.

A computation shows that $I = \langle x^3 - yz, y^2 - xz, z^2 - x^2y\rangle$ is contained in the ideal $\ker\varphi$ (one can show that the two ideals are equal!). This means that $\mathbb{C}[x,y,z]/\ker\varphi \subseteq \mathbb{C}[x,y,z]/I$. But by the homomorphism theorem one has $\mathbb{C}[x,y,z]/\ker\varphi \cong \mathbb{C}[t^3,t^4,t^5]$, thus $\mathbb{C}[x,y,z]/I$ contains the ring $\mathbb{C}[t^3,t^4,t^5]$.

But this ring contains the ring $\mathbb{C}[t^3]$, which has infinite dimension as a $\mathbb{C}$-vector space. So the cardinality of $V(x^3 - yz, y^2 - xz, z^2 - x^2y)$ is infinity.

Problem 4. (a) Fix a monomial order on $\mathbb{N}^3$ and let $K = \mathbb{C}$. Are the polynomials $P_1 = x^3 - yz$, $P_2 = x^2y - z^3$ and $P_3 = y^2 - z^2$ a Gröbner basis with respect to this order?

(b) If not, then complete the polynomials to a Gröbner basis.

(c) Does the system of equations $P_1(x,y,z) = P_2(x,y,z) = P_3(x,y,z) = 0$ have a solution? (Try to answer this question without actaully calculating one!)

Solution. (a) Define a linear order by $\lambda = (\frac{\sqrt{3}}{2}, \frac{3\sqrt{2}}{4}, 1)$. This is a linear order because the components $\sqrt{2}, \sqrt{3}, 1$ are in $\mathbb{R}_+$ and they are $\mathbb{Q}$-linearly independent (see this by assuming that there exists a dependence relation

$$q_1\sqrt{3} + q_2\sqrt{2} + q_3 = 0\,,$$

with $q_i \in \mathbb{Q}$ (we absorbed the fractions into q_i!). Clearing denominators, we may assume that $q_i \in \mathbb{Z}$. Assume that $q_2 \neq 0$ (the argument goes similar for q_1, q_3), then we may write $\sqrt{2} = \frac{-q_3 - \sqrt{3}q_1}{q_2}$. Squaring this equation yields $2 = \frac{q_3^2 + 3q_1^2 + 2\sqrt{3}q_1q_3}{q_2^2}$. Now rewrite this equation in the form $2q_1q_3\sqrt{3} = \ldots \in \mathbb{Q}$. This can only hold if either $q_1 = 0$ or $q_3 = 0$. Plugging $q_1 = 0$ into the original equation yields $\sqrt{2} \in \mathbb{Q}$, which is a contradiction. Similarly, $q_3 = 0$ would mean that $\sqrt{\frac{2}{3}} \in \mathbb{Q}$, also a contradiction. This shows that λ defines a linear order.)
Then $\mathrm{lm}_\lambda(P_1) = x^3$, $\mathrm{lm}_\lambda(P_2) = z^3$, and $\mathrm{lm}_\lambda(P_3) = y^2$. Then $S_{12} = x^5y - yz^4 = x^2yP_1 - yzP_2$, thus $\overline{S_{12}}^{(P_1,P_2,P_3)} = 0$. Similarly: $S_{23} = x^2y^3 - z^5 = z^2P_2 + x^2yP_3$ and $S_{13} = -y^3z + x^3 + x^3z^2 = z^2P_1 - yzP_2$. Thus by Buchberger's criterion, the P_i form a Gröbner basis with respect to λ.

Note: One can show that if the leading monomials (with respect to a chosen monomial order) of a collection of polynomials $P_1, \ldots, P_k$ do not have any nontrivial factors in common, then the $P_1, \ldots, P_k$ already form a Gröbner basis with respect to the chosen monomial order.

(b) If we had chosen another monomial order, e.g. *lex* with $z > y > x$, then we see that $\mathrm{lm}_{lex}(P_1) = yz$, $\mathrm{lm}_{lex}(P_2) = z^3$ and $\mathrm{lm}_{lex}(P_3) = z^2$. Using the notation from the lecture, we have $F_0 = \{P_1, P_2, P_3\}$. Then we get the S-polynomials: $S_{12} = x^3y^2 - x^2y^2$, $S_{13} = x^3z - y^3$ and $S_{23} = x^2y - x^3y$. One immediately sees that $S_{12} = yS_{23}$. Thus $F_1 = \{P_1, P_2, P_3, P_4 = S_{13}, P_5 = S_{23}\}$. Calculating S-polynomials again, we only get one new one: $S_{15} = x^6 - x^5$.

Calculating S-polynomials again, we find that all of them reduce to 0 by division through F_1. Thus F_1 is a Gröbner basis with respect to lex.

(c) For this we have to determine whether $1 \in \langle P_1, P_2, P_3 \rangle$. Using the monomial order from (a), we easily see that 1 is not contained in this ideal and thus there is a solution of the system of polynomial equations. (One easily sees that $(0, 0, 0)$ is one of the solutions!)