MATH3195/5195M EXERCISE SHEET 4
SOLUTIONS

DUE: APRIL 2ND, 2024

Primary decomposition, Noether normalisation and the Nullstellensatz

Problem 1. (a) Let $I = \langle x^2, y^2, z^2 \rangle \cap \langle x+y \rangle \cap \langle x-y \rangle$ be an ideal in $R = \mathbb{R}[x,y,z]$. Is the given intersection of ideals a (minimal) primary decomposition of I? Explain!

(b) Let I be a *monomial ideal* in $\mathbb{Q}[x_1, \dots, x_n]$, that is, I is generated by monomials. Show that if R is generated by pure powers of a subset of the variables, then it is a primary ideal. Further show that if $r = r_1 r_2$ is a minimal generator of I, where r_1 and r_2 are relatively prime, then

$$I = (I + \langle r_1 \rangle) \cap (I + \langle r_2 \rangle) .$$

Remark: This yields an algorithm to compute primary decomposition of a monomial ideal!

Solution. (a)

The ideals $I_1 = \langle x-y \rangle$ and $I_2 = \langle x+y \rangle$ are both prime in $R = \mathbb{R}[x,y,z]$, since $R/I_1 \cong R/I_2 \cong \mathbb{R}[y,z]$ are integral domains.

The ideal I_3 is $\langle x,y,z \rangle$-primary (the radical of I_3 is clearly $\langle x,y,z \rangle$ and by Proposition of the lecture I_3 is then primary). Thus $I = I_1 \cap I_2 \cap I_3$ is a primary decomposition.

However, it is not minimal, since $I_1 \cap I_2 = \langle x^2 - y^2 \rangle$ is contained in I_3.

(b)

The first assertion immediately follows from the definition of primary: I is generated by some subset of the elements $x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$ for $\alpha_i \geq 1$. Look at $\mathbb{Q}[x_1, \dots, x_n]/I$: this is non-zero, since I is a proper ideal, and every element $\bar{f}$ is nilpotent (if $\alpha = \sum_{i=1}^n \alpha_i$, then $\bar{f}^\alpha = \bar{0}$).

For the second assertion note that I and $(I + \langle r_1 \rangle) \cap (I + \langle r_2 \rangle)$ contain the same monomials.

A monomial m is contained in $(I + \langle r_i \rangle)$ if and only if $m \in I$ or r_i divides m. Since r_1 and r_2 are relatively prime, we have $m \in (I + \langle r_1 \rangle) \cap (I + \langle r_2 \rangle)$ if and only if $m \in I$ or $r_1 r_2$ divides m. This is equivalent to saying that $m \in I$.

Problem 2. (a) Let $R = \mathbb{R}[x,y]/\langle x^5 - y^3\rangle$. Show that $t = \frac{y}{x}$ and $u = \frac{x^2}{y}$ are integral over R. What are the R-module generators of $R[t]$ and $R[u]$?

(b) Let $f = x^3 + y^2$. Show that f is integral over $\mathbb{Q}[x^6, y^2]$.

(c) Let R be a unique factorisation domain, that is, R is an integral domain and every element in R can be written as a product of irreducible elements, unique up to order and multiplication with units. Show that every integral element of the form $\frac{x}{y}$, $x, y \in R$ is already contained in R. (*Remark*: this shows that R is *integrally closed* in its field of fractions).

Solution. (a)

In order to see that t and u are integral over R, we have to find some integral equations they satisfy.

The easiest way to find those, is to use the isomorphism of rings $R \cong \mathbb{R}[z^3, z^5]$ under which $t = z^2$ and $u = z$. Then it is clear that $t^3 = z^6$ and $u^3 = z^3$ and thus $t^3 - x^2 = 0$ and $u^3 - x = 0$ are the desired integral equations in R.

From the integral equations it follows that $R[t] = R + Rt + Rt^2$ and $R[u] = R + Ru + Ru^2$ as R-modules.

(b)

We find a monic polynomial F in $(\mathbb{Q}[x^6, y^2])[t]$ that has f as a root. Compute $f^2 = x^6 + 2x^3y^2 + y^4$. This gives us the relation $f^2 - y^4 - x^6 = 2x^3y^2$. Squaring again yields

$$f^4 + y^8 + x^{12} - (2y^2 - 2x^6)f^2 + x^6y^4 = 4x^6y^4\ ,$$

so the integral relation for f is given by

$$F(t) = t^4 - (2y^2 - 2x^6)t^2 + y^8 + x^{12} + x^6y^4 - 4x^6y^4 \in \mathbb{Q}[x^6, y^2, t].$$

(c)

Let $\frac{x}{y}$ with $x, y \in R$ be integral over R. By assumption on R we may suppose that x, y are relatively prime. Now $\frac{x}{y}$ satisfies an equation of integral dependence:

$$\left(\frac{x}{y}\right)^n + a_{n-1}\left(\frac{x}{y}\right)^{n-1} + \cdots + a_0 = 0\ ,$$

for some $a_i \in R$. Multiplying with y^n yields the equation

$$x^n = -y(a_{n-1}x^{n-1} + \cdots + a_0y^n)\ .$$

This means that x^n is a multiple of y. If y were not a unit, this would yield a contradiction to our assumption that x is relatively prime to y. Therefore y is a unit and $\frac{x}{y}$ is an element of R.

Problem 3. Decompose $X := V((x^2y - xy^2)(x+y)) \subseteq \mathbb{A}^2_{\mathbb{R}}$ into irreducible components, that is, write X as a union of $V(f_i)$, where the f_i are irreducible polynomials. Same question for $X \subseteq \mathbb{A}^2_{\mathbb{F}_2}$, where $\mathbb{F}_2$ denotes the field with two elements.

First note that X is a hypersurface and $(x^2y - xy^2)(x+y) = xy(x-y)(x+y)$.

Since $x+y$, $x-y$, x, y, are all irreducible in $\mathbb{R}[x,y]$ it follows that the minimal primary decomposition of $\sqrt{I} = I = \langle xy(x-y)(x+y)\rangle$ is $\langle x+y\rangle \cap \langle x-y\rangle \cap \langle x\rangle \cap \langle y\rangle$.

Thus $V(I) = V(x+y) \cup V(x-y) \cup V(x) \cup V(y)$ is a union of four lines meeting at the origin.

When considering the same $I \subseteq \mathbb{F}_2[x,y]$, one sees that $x-y = x+y$ and thus $(x-y)(x+y) = x^2 - y^2 = (x+y)^2$. So I is not radical anymore and the minimal primary decomposition is $I = \langle x+y\rangle^2 \cap \langle x\rangle \cap \langle y\rangle$ (cf. the lecture).

Then $V(I) = V(x+y) \cup V(x) \cup V(y)$ has three irreducible components, three "lines" meeting at the origin.

Problem 4. Sketch the following affine algebraic sets (you may use a computer algebra program for this!)

(a) $V(y^2 - x^5) \subset \mathbb{A}^2_{\mathbb{R}}$
(b) $V((x^2+y^2)^2 + 4x(x^2+y^2) - 4y^2) \subset \mathbb{A}^2_{\mathbb{R}}$
(c) $V(x^2+y^2-1) \subset \mathbb{A}^3_{\mathbb{R}}$,
(d) $V(x^3 + x^2z^2 - y^2) \subset \mathbb{A}^3_{\mathbb{R}}$
(e) $V(x^4y^2 - x^2y^4 - x^4z^2 + y^4z^2 + x^2z^4 - y^2z^4) \subset \mathbb{A}^3_{\mathbb{R}}$

Solution. (a) This is a curve in the plane, a so-called *higher cusp*.

(b) This is a *cardioid*, see e.g. `https://en.wikipedia.org/wiki/Cardioid`.

(c) This is a cylinder with radius 1 about the origin in $\mathbb{R}^3$.

(d) This surface is called *kolibri* and often used for counterexamples about limits of tangent spaces. In Fig. 1 is a visualization (made with POV-ray).

(e) One can factor the polynomial $x^4y^2 - x^2y^4 - x^4z^2 + y^4z^2 + x^2z^4 - y^2z^4$ as $(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)$, so its vanishing set consists of 6 hyperplanes meeting at the origin. This is the so-called A_3-hyperplane arrangement, see Fig. 2.

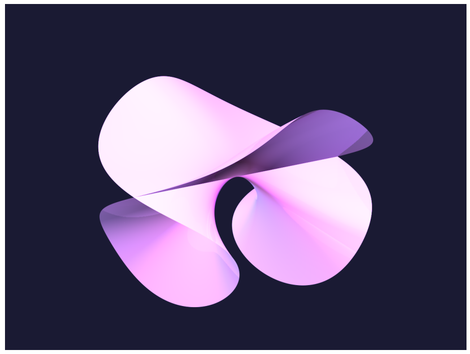

FIGURE 1. Kolibri: $V(x^2 - y^2z^2 - y^3)$

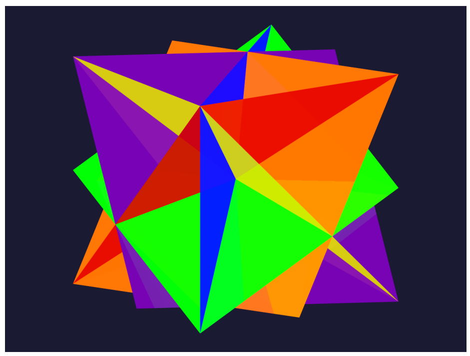

FIGURE 2. The hyperplane arrangement $V((x^2 - y^2)(x^2 - z^2)(y^2 - z^2))$

Problem 5. Let $F = (x^2 - y^3)^2 - (z^2 - y^2)^3$ be a polynomial in $\mathbb{R}[x, y, z]$.

(a) Sketch $V(F) \subset \mathbb{A}^3_{\mathbb{R}}$.
(b) Let $J_F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$ be the Jacobian ideal of F. Find $V(J_F)$ and sketch it.
(c) Is J_F radical?

Solution. (a)

This surface is called *Daisy*. For a visualization see Fig. 3 and more about the construction is here `http://www.ams.org/journals/bull/2010-47-03/S0273-0979-10-01295-4/S0273-0979-10-01295-4.pdf`.

(b)

The Jacobian ideal $J_F = \langle -4xy^3 + 4x^3, 12y^5 - 12y^3z^2 + 6yz^4 - 6x^2y^2, -6y^4z + 12y^2z^3 - 6z^5 \rangle$. Here the easiest way to find $V(J_F)$ is by noting that $V(J_F) = V(\sqrt{J_F})$.

Now using a computer algebra system (such as Singular or Maple) yields that $\sqrt{J_F} = \langle y^2 - z^2, yz^2 - x^2, z^4 - x^2y \rangle$. Setting the first equation equal to 0 we see that either $y = z$ or $y = -z$.

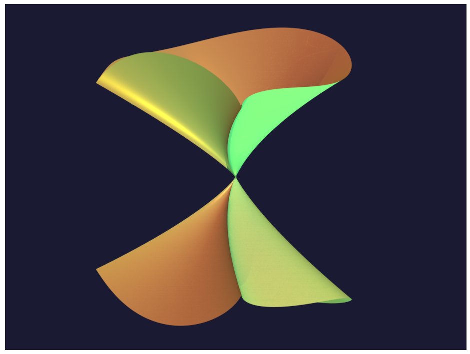

FIGURE 3. $V((x^2-y^3)^2-(z^2-y^2)^3)$

In the first case (plugging $y=z$ into the two remaining generators) we get the system $y^3-x^2=0$ and $y^4-x^2y=0$, which yields the curve $V(y-z, y^3-x^2)$, or parametrized $\{(t^3,t^2,t^2)$ for $t\in\mathbb{R}\}$.

For $y-z$ we similarly obtain the second component of $V(J_F)$, namely $V(y+z, x^2+z^3)$, which is parametrized by $\{(t^3,t^2,-t^2)$ for $t\in\mathbb{R}\}$. So $V(J_F)$ is the union of two space curves.

(c)

J_F is not radical, since for example $y^2-z^2\in\sqrt{J_F}$ but not in J_F (get that y^2-z^2 is contained in the radical from the calculation of $\sqrt{J_F}$ above, then it is easy to check that it is not contained in J_F).

Problem 6. The image of a non-constant complex polynomial map $f:\mathbb{C}^2\to\mathbb{C}^3$ is a hypersurface. Let $f(s,t)=(s^3t^3,s^2,t^2)$.

(a) Find an irreducible polynomial map $F:\mathbb{C}^3\to\mathbb{C}$ such that $\mathrm{Im}(f)\subset V(F)$. (Use coordinates (x,y,z).)
(b) Let again $J_F=\left\langle \frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right\rangle$ be the Jacobian ideal of F. Find a minimal primary decomposition of J_F and its associated primes. (Hint: Ensure J_F is simplified as much as possible and try to guess the primary components!)
(c) Hence show that J_F has an embedded prime and two isolated primes.

Solution. (a)

Let $F(x,y,z)=x^2-y^3z^3$. Then

$$F(s^3t^3,s^2,t^2)=(st)^6-s^6t^6=0$$

so $\mathrm{Im}(f)\subseteq V(f)$.

(b)

Direct calculation shows that $J_F = \langle x, y^2z^3, y^3z^2 \rangle$. First we look at $V(J_F)$: it is easy to see that $V(J_F) = V(x,y) \cup V(x,z)$ is a union of two lines.

So the primary decomposition of $\sqrt{J_F} = \langle x, y \rangle \cap \langle x, z \rangle$. Starting from this we see that $J_F \subseteq \langle x, y^2 \rangle \cap \langle x, z^2 \rangle = \langle x, y^2z^2 \rangle$.

In order to get the third powers, we intersect with $\langle x, y^3, z^3 \rangle$ and by a direct calculation that $J_F = \langle x, y^3, z^3 \rangle \cap \langle x, z^2 \rangle \cap \langle x, y^2 \rangle$.

All the ideals on the right hand side are primary (see e.g., that in $\mathbb{C}[x,y,z]/J$, where J is one of the three ideals, every nonconstant element is nilpotent).

Moreover, their radicals are distinct and none is contained in the intersection of the other two. Thus the minimal primary decomposition consists of the three ideals and their associated primes are $\mathfrak{p}_1 = \langle x, y, z \rangle$, $\mathfrak{p}_2 = \langle x, y \rangle$, and $\mathfrak{p}_3 = \langle x, z \rangle$.

(c)

Using the results of (b), by definition $\mathfrak{p}_1$ is embedded and $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are both isolated.