MATH3195/5195M EXERCISE SHEET 3
SOLUTIONS

DUE: MARCH 11, 2024

Problem 1. (a) Let $R = \mathbb{Q}[[x,y]]$ and let $J = \langle xy + y^3, x + x^2y, xy + 3y, x^4 - 5y^2 + x^2y \rangle$ be an ideal in R. Show that J is minimally generated by two elements in R.

(b) Let $R = K[t]$ and consider $M = K[t, t^{-1}]$ as R-module and let $I = tR$ be an ideal in R. Show that $M = IM$ but $M \neq 0$. Why does this example not contradict Nakayama's lemma?

Solution. (a) Use the Nakayama lemma to show that $J = \mathfrak{m} = \langle x, y \rangle$: the lemma says that if for a finitely generated R-module M and a submodule $N \subseteq M$ and an ideal $I \subseteq J(R)$ one has the equality $M = N + IM$, then $N = M$.

In this example R is a local ring with maximal ideal $\mathfrak{m}$, thus we have $\mathfrak{m} = J(R)$. Clearly $J \subseteq \mathfrak{m}$.

We show that $\mathfrak{m} = J + \mathfrak{m}\mathfrak{m}$, then by Nakayama's lemma it follows that $\mathfrak{m} = J$: first calculate $\mathfrak{m}^2 = \langle x^2, xy, y^2 \rangle$. Then

$$J + \mathfrak{m}^2 = xy + y^3, x + x^2y, xy + 3y, x^4 - 5y^2 + x^2, x^2, xy, y^2 \rangle = \langle x, 3y \rangle \ ,$$

which is equal to $\mathfrak{m}$, since $3 \in \mathbb{Q}^*$.

For the minimal number of generators we can use the third version of Nakayama's lemma: $(R, \mathfrak{m})$ is a local ring and $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{Q}x \oplus \mathbb{Q}y$ as a $R/\mathfrak{m} \cong \mathbb{Q}$-vector space. So x and y form a basis of this vector space and Nakayama's lemma allows to conclude that hey are a minimal set of generators of $\mathfrak{m}$.

(b) First calculate IM: these are all elements of the form $f(t)tg(t, t^{-1})$, where $f(t) \in R$, $g(t, t^{-1}) \in M$.

Clearly this element is again a polynomial in t and t^{-1}, so is contained in M.

On the other hand, it is also clear that $M \subseteq IM$, since any element $g(t, t^{-1})$ of M can be written as $t(t^{-1}g(t, t^{-1}))$, with $(t^{-1}g(t, t^{-1})) \in M$.

Thus we have $IM = M$.

There are various conditions of Nakayama's lemma that are not satisfied: I is not a subset of $J(R) = \langle 0 \rangle$. Also, M is not finitely generated as an R-module (as R-module, $M = R + Rt^{-1} + Rt^{-2} + \cdots$. Note that this is not a *direct* sum!).

Problem 2. Prove the isomorphism theorems for modules.

Solution. Note that solution includes the proofs of all isomorphism theorems for your reference.

(1) Use the notation from the lecture: let $\phi : M \to N$ be an R-module homomorphism. Define a map $\tilde{\phi} : M/\ker\phi \to \mathrm{im}\phi$ by

$$\tilde{\phi}(m + \ker\phi) = \phi(m).$$

- (**Well defined**) If $m + \ker\phi = m' + \ker\phi$, then $m - m' \in \ker\phi$. So $\tilde{\phi}(m + \ker\phi) = \phi(m) - \phi(m - m') = \phi(m') = \tilde{\phi}(m' + \ker\phi)$.

- (R-**homomorphism**) $\tilde{\phi}(r(m + \ker\phi) + s(n + \ker\phi)) = \tilde{\phi}((rm + sn) + \ker\phi) = \phi(rm + sn) = r\phi(m) + s\phi(n) = r\tilde{\phi}(m + \ker\phi) + s\phi(n + \ker\phi)$.

- (**Injective**) If $\phi(m + \ker\phi) = 0$ then $\phi(m) = 0$, so $m \in \ker\phi$ and $m + \ker\phi = 0$.

- (**Surjective**) Clear.

So $\tilde{\phi}$ is an R-isomorphism.

(2) Here assume that $M \supseteq N \supseteq L$ are R-modules. Define a map $\phi : M/L \to M/N$ by

$$\phi(m + L) = m + N.$$

- (**Well defined**) If $m + L = m' + L$, then $m - m' \in L \subset N$, so $m + N = m' + N$.

- (R-**homomorphism**) $\phi(r(m + L) + r'(m' + L)) = \phi((rm + r'm') + L) = (rm + r'm') + N = r(m + N) + r'(m' + N) = r\phi(m + L) + r'\phi(m' + L)$.

- (**Kernel**) $m + L \in \ker\phi \iff \phi(m + L) = 0 \iff m + N = 0 \iff m \in N \iff m + L \in N/L$.

- (**Image**) Clearly ϕ is surjective.

So by (i), $\tilde{\phi}$ defines an isomorphism $(M/L)/(N/L) \cong (M/N)$.

(3) Define $\phi : M \to (M + L)/L$ by

$$\phi(m) \to m + L$$

. (Note that $m \in M \subset M + L$).

- (R-**homomorphism**) $\phi(rm + sn) = (rm + sn) + L = r(m + L) + s(n + L) = r\phi(m) + s\phi(n)$.

- (**Kernel**) $m \in \ker\phi \iff \phi(m) = 0 \iff m + L = 0 \iff m \in L \iff m \in M \cap L$.

- (**Image**) Let $(m + \ell) + L$ in $(M + L)/L$. Then $(m + \ell) - m \in L$, so $m + L = (m + \ell) + L$. Now $\phi(m) = m + L = (m + \ell) + L$, so ϕ is surjective.

By (i), $\tilde{\phi}$ defines an isomorphism $M/(M \cap L) \cong (M + L)/L$.

Problem 3. Prove the 3×3-lemma: Let R be a ring. Assume that

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\alpha} & A_2 & \xrightarrow{\alpha'} & A_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
0 & \longrightarrow & B_1 & \xrightarrow{\beta} & B_2 & \xrightarrow{\beta'} & B_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \\
0 & \longrightarrow & C_1 & \xrightarrow{\gamma} & C_2 & \xrightarrow{\gamma'} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

is a commutative diagram of R-modules and all columns and the middle row is exact. Show that the top row is exact if and only if the bottom row is exact.

Solution. (a) Use the snake lemma: First assume that the top row is exact. Then since the diagram commutes and the second row is exact, the snake lemma yields the exact sequence:

$$0 \to \ker(f_1) \to \ker(f_2) \to \ker(f_3) \to \operatorname{coker}(f_1) \to \operatorname{coker}(f_2) \to \operatorname{coker}(f_3) \to 0\,.$$

Since the i-th column is exact, $\ker(f_i) = 0$ and $\operatorname{coker}(f_i) \cong C_i$ for $i = 1, 2, 3$. Thus the above exact sequence becomes the short exact sequence

$$0 \to C_1 \to C_2 \to C_3 \to 0\,,$$

which had to be shown.

Similarly, if the bottom row is exact, the snake lemma gives us the exact sequence

$$0 \to \ker(g_1) \to \ker(g_2) \to \ker(g_3) \to \operatorname{coker}(g_1) \to \operatorname{coker}(g_2) \to \operatorname{coker}(g_3) \to 0\,.$$

Since the i-th column is exact, $\ker(g_i) = A_i$ and $\operatorname{coker}(g_i) \cong 0$ for $i = 1, 2, 3$. This simplifies to the short exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0\,,$$

which had to be shown.

Problem 4. (Localisation of a module) Let R be a ring and $A \subset R$ be multiplicatively closed. Let M be an R-module. Assume we know that $(m, a) \sim (n, b)$ if and only if $mbc = nac$ for some $c \in A$ defines an equivalence relation on $M \times A$.

(a) Writing $A^{-1}M$ for the set of equivalence classes of $\sim$, and $\frac{m}{a}$ for the class containing (m, a), show that the operation

$$\frac{m}{a} + \frac{n}{b} = \frac{bm + an}{ab}$$

is well defined and hence that $A^{-1}M$ is an abelian group.

(b) By defining an appropriate multiplication rule, show that $A^{-1}M$ has the structure of an $A^{-1}R$-module.

Solution. (a) It suffices to prove the result for $\frac{m'}{a'} = \frac{m}{a}$. Then we have $c \in A$ such that $m'ac = ma'c$. Now $\frac{m'}{a'} + \frac{n}{b} = \frac{bm'+a'n}{a'b}$, but

$$(bm' + a'n)(ab)c = (m'ac)b^2 + aa'bcn = (ma'c)b^2 + aa'bcn = (bm + an)(a'b)c$$

so each sum is equal to the same class, so the addition is well defined. Associativity is clear by associativity of M, the inverse of $\frac{m}{a}$ is $\frac{-m}{a}$ and the identity is $\frac{0}{1}$.

(b) We define $\frac{r}{a}\frac{m}{b} = \frac{rm}{ab}$. It is easy to then check that this is a module.

Problem 5. Let R be a ring and $A \subset R$ be multiplicatively closed.

(a) Suppose that $\phi : M \to N$ is a homomorphism of R modules. Show ϕ induces an $A^{-1}R$-homomorphism $A^{-1}M \to A^{-1}N$.
(b) Suppose $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules. Show that $0 \to A^{-1}L \to A^{-1}M \to A^{-1}N \to 0$, with the induced maps from (i), is an exact sequence of $A^{-1}R$-modules. (*Remark*: This means that localization is an exact functor from the category of R-modules to the category of $A^{-1}R$-modules)

Solution. (a) We define $\tilde{\phi} : A^{-1}M \to A^{-1}N$ by

$$\tilde{\phi}(\frac{m}{a}) = \frac{\phi(m)}{a}.$$

- (**Well defined**) Suppose $\frac{m'}{a'} = \frac{m}{a}$, then there is some $c \in A$ such that $ma'c = m'ac$. Now $\phi(m)a'c = \phi(ma'c) = \phi(m'ac) = \phi(m')ac$, so $\frac{\phi(m)}{a} = \frac{\phi(m')}{a'}$.

- ($A^{-1}R$**-hom**) For $r \in R$, $m, n \in M$ and $a, b, c \in A$ we have

$$\begin{aligned}\tilde{\phi}\left(\frac{r}{a}\frac{m}{b} + \frac{n}{c}\right) &= \tilde{\phi}\left(\frac{rmc + nab}{abc}\right) \\ &= \frac{\phi(rmc + nab)}{abc} \\ &= \frac{r\phi(m)c + \phi(n)ab}{abc} \\ &= \frac{r}{a}\frac{\phi(m)}{b} + \frac{\phi(n)}{c} \\ &= \frac{r}{a}\tilde{\phi}\left(\frac{m}{b}\right) + \tilde{\phi}\left(\frac{n}{c}\right).\end{aligned}$$

(b) Let $\phi : L \to M$, $\psi : M \to N$ be the above maps.

- (**Exact at** $A^{-1}L$) We show that $\tilde{\phi}$ is injective, so suppose that $\tilde{\phi}(\frac{\ell}{a}) = \frac{\phi(\ell)}{a} = 0$. Then there is some $c \in A$ such that $\phi(\ell)c = \phi(\ell c) = 0$. Since ϕ is injective, we have $\ell c = 0$ and hence $\frac{\ell}{a} = 0$.

- **(Exact at** $A^{-1}M$) Firstly, $\tilde{\psi}(\tilde{\phi}(\frac{\ell}{a})) = \frac{\psi(\phi(\ell))}{a} = 0$ since the original sequence is exact. Thus $\mathrm{im}\tilde{\phi} \subset \ker\tilde{\psi}$. Now suppose $\frac{m}{a} \in \ker\tilde{\psi}$, so $\tilde{\psi}(\frac{m}{a}) = \frac{\psi(m)}{a} = 0$. Thus there is some $c \in A$ such that $\psi(m)c = \psi(mc) = 0$, so $mc \in \ker\psi = \mathrm{im}\phi$ and we can write $mc = \phi(\ell)$ for some $\ell \in L$. But now $\tilde{\phi}(\frac{\ell}{ac}) = \frac{\phi(\ell)}{ac} = \frac{mc}{ac} = \frac{m}{a}$, so $\ker\tilde{\psi} \subset \mathrm{im}\tilde{\phi}$.

- **(Exact at** $A^{-1}N$) We show $\tilde{\psi}$ is surjective, so suppose $\frac{n}{a} \in A^{-1}N$. Now since ψ is surjective we have $m \in M$ such that $\psi(m) = n$. Then $\tilde{\psi}(\frac{m}{a}) = \frac{\psi(m)}{a} = \frac{n}{a}$.