MATH3195/5195M EXERCISE SHEET 2
SOLUTIONS

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Problem 1. (a) Show that $\mathbb{R}[x,y]/(x^3-y^2)$ is isomorphic to $\mathbb{R}[t^2,t^3]$. [Hint: First homomorphism theorem. First show that $f(x,y) = x^3-y^2$ is in the kernel of the map φ as defined in the lecture. In order to see that $(f(x,y))$ is the full kernel, you may use the fact, that the kernel of φ is generated by elements of the form $x^a y^b - x^{a'} y^{b'}$, where $a, a', b, b' \in \mathbb{N}$. This fact can be proved using Gröbner bases methods]

(b) Is (x^3-y^2) a prime ideal in $\mathbb{R}[x,y]$? Explain!

Solution. (a) Define $\varphi : \mathbb{R}[x,y] \to \mathbb{R}[t]$ by $\varphi(x) = t^2$ and $\varphi(y) = t^3$. The image of φ is the subring of $\mathbb{R}[t]$ generated by t^2 and t^3, that is, the ring $\mathbb{R}[t^2,t^3]$. Then by the first homomorphism theorem, $\mathrm{im}(\varphi) = \mathbb{R}[t^2,t^3] \cong \mathbb{R}[x,y]/\ker\varphi$. It remains to determine $\ker\varphi$. The element $f(x,y) = x^3-y^2$ is in $\ker\varphi$ since $\varphi(f(x,y)) = f(t^2,t^3) = (t^2)^3 - (t^3)^2 = 0$. Now use the hint, which says that $\ker\varphi = (x^a y^b - x^{a'} y^{b'}$, where a, a', b, b' are some integers $\in \mathbb{N}$). Assume that $g(x,y) = x^a y^b - x^{a'} y^{b'}$ is in $\ker\varphi$, that means that $t^{2a+3b} - t^{2a'+3b'} = 0$. So we are looking for all integer linear combinations such that $2a+3b = 2a'+3b'$, or $2(a-a') = 3(b'-b)$. We may assume w.l.o.g. that $a > a'$ and hence $b' > b$ (if $a = a'$ we would get $b = b'$ and the binomial would be 0). Since $a - a'$ and $b' - b$ are integers and 2 and 3 are coprime, the above equation implies that $3|(a-a')$ and $2|(b'-b)$. Thus $a - a' = 3k$, which implies that $b' - b = 2k$ for some $k \in \mathbb{N}_{>0}$. Thus any $g(x,y)$ in $\ker\varphi$ is of the form $x^{a'+3k}y^{b'-2k} - x^{a'}y^{b'} = x^{a'}y^{b'-2k}(x^{3k} - y^{2k})$. Since $x^{3k} - y^{2k} = (x^3)^k - (y^2)^k = (x^3-y^2)(\sum_{i=0}^{k-1} x^{3i}y^{2(k-1-i)})$, one sees that $f(x,y)|g(x,y)$, which means that $g(x,y) \in (f(x,y))$. Thus $\ker\varphi = (x^3-y^2)$ and the first isomorphism theorem shows that $\mathbb{R}[t^2,t^3] \cong \mathbb{R}[x,y]/(x^3-y^2)$.

(b) Since $\mathbb{R}[t^2,t^3]$ is a subring of the integral domain $\mathbb{R}[t]$, it is itself an integral domain (if we had $a(t^2,t^3)b(t^2,t^3) = 0$, then since both $a, b \in \mathbb{R}[t]$, this implies that either a or b is 0). By (a) $\mathbb{R}[x,y]/(x^3-y^2) \cong \mathbb{R}[t^2,t^3]$, hence $\mathbb{R}[x,y]/(x^3-y^2)$ is an integral domain. By the theorem from the lecture (x^3-y^2) is a prime ideal in $\mathbb{R}[x,y]$.

Problem 2. (a) Show that the ideal $(x^4 - 5x^3 + 7x^2 - 5x + 6, x^4 + 2x^2 + 1, x^4 - 2x^3 + x^2 - 2x)$ in $\mathbb{R}[x]$ is maximal.

(b) Let R be a ring such that every element satisfies $x^n = x$ for some $n > 1$ (here the integer n depends on x). Show that every prime ideal in R is maximal.

Solution. (a) If $(f(x), g(x), h(x))$ is an ideal in $K[x]$, where K is a field, then one can see that $(f(x), g(x), h(x)) = (\gcd(f, g, h))$.
We first calculate the factorizations of the polynomials $f(x) = x^4 - 5x^3 + 7x^2 - 5x + 6 = (x-2)(x-3)(x^2+1)$, $g(x) = x^4 + 2x^2 + 1 = (x^2+1)^2$ and $h(x) = x^4 - 2x^3 + x^2 - 2x = x(x-2)(x^2+1)$ into irreducible polynomials in $\mathbb{R}[x]$ (use e.g. rational root test). Thus we see that the gcd of $f(x), g(x), h(x)$ is x^2+1 and $(f(x), g(x), h(x)) = (x^2+1)$. But x^2+1 is irreducible in $\mathbb{R}[x]$, thus $\mathbb{R}[x]/(x^2+1)$ is a field. This means that (x^2+1) is a maximal ideal in $\mathbb{R}[x]$.
(b) Let $\mathfrak{p} \subseteq R$ be a prime ideal and let $x \in R \backslash \mathfrak{p}$ with $x^n = x$ for some $n > 1$. Since $\mathfrak{p}$ is prime, $R/\mathfrak{p}$ is an integral domain and $\bar{x} \neq \bar{0}$ is a nonzero-divisor. Then from the equation $\bar{x}^n - \bar{x} = \bar{x}(\bar{x}^{n-1} - \bar{1}) = \bar{0}$ we can cancel $\bar{x}$ and obtain $\bar{x}^{n-1} = \bar{1}$. But this means that $\bar{x}\bar{x}^{n-2} = \bar{1}$, that is, $\bar{x}^{n-2}$ is a multiplicative inverse of $\bar{x}$. Thus any element $\bar{x} \neq \bar{0} \in R/\mathfrak{p}$ is invertible, which implies that $R/\mathfrak{p}$ is a field. But then (as shown in the lecture) $\mathfrak{p}$ is a maximal ideal.

Problem 3. (a) Consider $K[x, y, z]$ and order all monomials of degree less than or equal to 3 with respect to the following monomial orders: (i) $<_{lex}$, (ii) $<_{deglex}$, (iii) $<_\lambda$, where λ is a suitable linear form $\lambda : \mathbb{R}^3 \to \mathbb{R}$.
(b) Determine leading monomial and coefficient of the polynomial $f = x^4 + z^5 + x^3z + yz^4 + x^2y^2$ with respect to the momomial orders from (a).

Solution. (a) All monomials of degree ≤ 2 are: $1, x, y, z, x^2, xy, xz, yz, y^2, z^2$.The orders are:
(i) $1 <_{lex} z <_{lex} z^2 <_{lex} y <_{lex} yz <_{lex} y^2 <_{lex} x <_{lex} xz <_{lex} xy <_{lex} x^2$
(ii) $1 <_{deglex} z <_{deglex} y <_{deglex} x <_{deglex} z^2 <_{deglex} yz <_{deglex} y^2 <_{deglex} xz <_{deglex} xy <_{deglex} x^2$.
(iii) We have to choose a λ with $\mathbb{Q}$-linearly independent entries. Take e.g. $\lambda = (1, \sqrt{2}, \sqrt{5})$. Then
$1 <_\lambda x <_\lambda y <_\lambda z <_\lambda x^2 <_\lambda xy <_\lambda y^2 <_\lambda xz <_\lambda yz <_\lambda z^2$.
(b) (i) $lm_{lex}(f) = x^4$ and $lc_{lex}(f) = 1$, (ii) $lm_{deglex}(f) = yz^4$ and $lc_{lex}(f) = 1$, (ii) with λ from above $lm_\lambda(f) = z^5$ and $lc_\lambda(f) = 1$.

Problem 4. Let R be a ring. Show that R is local if and only if the nonunits of R form a maximal ideal.

Solution. Let R be local, that is, there is a unique maximal ideal $\mathfrak{m} \subseteq R$. Denote $S = \{$ nonunits of $R\}$. We have to show that S is an ideal. Let $s, t \in S$. Then $\langle s \rangle + \langle t \rangle$ is an ideal and clearly $\langle s \rangle \subseteq \mathfrak{m}$ and $\langle t \rangle \subseteq \mathfrak{m}$. But this implies that $s - t \in \mathfrak{m}$. If $s \in S$ and $r \in R$, then $rs \in \mathfrak{m}$ since $s \in \mathfrak{m}$, thus S is an ideal in R. If $S \subsetneq \mathfrak{m}$, then there would exist a unit in $\mathfrak{m}$ (by definition of S). But this would mean that $\mathfrak{m} = R$, contradiction to

the fact that $\mathfrak{m}$ is a proper ideal of R.
For the other direction, assume that S is a maximal ideal in R.This means that S is a proper ideal of R. Let M be an arbitrary maximal ideal of R. Then every element of M has to be a non-unit of R (since M is supposed to be proper). This implies that $M \subseteq S$ and by maximality, $M = S$.

Problem 5. Let I be an ideal of R and A be a multiplicatively-closed subset of R. Show that:

(a) $A^{-1}I$ is an ideal of $A^{-1}R$;
(b) $\frac{x}{a} \in A^{-1}I$ if and only if there is some $b \in A$ with $xb \in I$;
(c) $A^{-1}I = A^{-1}R$ if and only if $I \cap A \neq \emptyset$;
(d) localization commutes with quotients, that is

$$A^{-1}R/A^{-1}I \cong \overline{A}^{-1}(R/I), \text{ where } \overline{A} = \{a + I : a \in A\}.$$

Solution. (a) Firstly, since $0 \in I$ and $1 \in A$ we have $\frac{0}{1} \in A^{-1}I$. Now suppose that $\frac{r}{a}, sb \in A^{-1}I$, then $\frac{r}{a} - \frac{s}{b} = \frac{rb-sa}{ab}$. Since $r, s \in I \subseteq R$ we have $rb - sa \in I$, and since A is multiplicatively closed we have $ab \in A$. Thus $\frac{r}{a} - \frac{s}{b} \in A^{-1}I$. Finally if $\frac{t}{c} \in A^{-1}R$ then $\frac{t}{c}\frac{r}{a} = \frac{tr}{ac}$, and again since $tr \in I$ and $ac \in A$ we have $\frac{t}{c}\frac{r}{a} \in A^{-1}I$, so $A^{-1}I \subseteq A^{-1}R$.
(b) If $\frac{x}{a} \in A^{-1}I$ then setting $b = 1$ gives the result. Conversely if $xb \in I$ then $\frac{xb}{ab} \in A^{-1}I$, but $\frac{xb}{ab} = \frac{x}{a}$.
(c) Suppose $A^{-1}I = A^{-1}R$, then $1_{A^{-1}R} = \frac{1}{1} \in A^{-1}I$. By (b), this implies that there is some $b \in A$ with $1 \cdot b \in I$, i.e. $I \cap A \neq \emptyset$. Conversely if $I \cap A \neq \emptyset$ then choose $a \in I \cap A$. Now $\frac{a}{a} \in A^{-1}I$, but $\frac{a}{a} = \frac{1}{1} = 1_{A^{-1}R}$, so $A^{-1}I = A^{-1}R$.
(d) Define $\phi : A^{-1}R \to \bar{A}^{-1}(R/I)$ by $\phi(\frac{r}{a}) = \frac{r+I}{a+I}$. It is easy to check that ϕ is well-defined. This map is a

homomorphism as given $\frac{r}{a}, \frac{s}{b} \in A^{-1}R$ we have

$$\begin{aligned}
\phi\left(\frac{r}{a}+\frac{s}{b}\right) &= \phi\left(\frac{rb+sa}{ab}\right) \\
&= \frac{(rb+sa)+I}{ab+I} \\
&= \frac{(r+I)(b+I)+(s+I)(a+I)}{(a+I)(b+I)} \\
&= \frac{r+I}{a+I}+\frac{s+I}{b+I} \\
&= \phi\left(\frac{r}{a}\right)+\phi\left(\frac{s}{b}\right), \\
\phi\left(\frac{r}{a}\frac{s}{b}\right) &= \phi\left(\frac{rs}{ab}\right) \\
&= \frac{rs+I}{ab+I} \\
&= \frac{(r+I)(s+I)}{(a+I)(b+I)} \\
&= \frac{r+I}{a+I}\frac{s+I}{b+I} \\
&= \phi\left(\frac{r}{a}\right)\phi\left(\frac{s}{b}\right), \text{ and} \\
\phi(1_{A^{-1}R}) &= \phi\left(\frac{1}{1}\right) \\
&= \frac{1+I}{1+I} \\
&= 1_{\bar{A}^{-1}(R/I)}.
\end{aligned}$$

Also ϕ is clearly surjective, and $\phi(\frac{r}{a}) = \frac{r+I}{a+I} = \frac{I}{1+I}$ iff there is some $c+I \in \bar{A}$ such that $(r+I)(1+I)(c+I) = I(a+I)(c+I)$, that is iff there is some $c \in A$ such that $rc \in I$. By part (b) this is iff $\frac{r}{a} \in A^{-1}I$, so $\ker\phi = A^{-1}I$. By the first isomorphism theorem the result follows.