MATH3195/M5195 EXERCISE SHEET 1
WITH SOLUTIONS

Problem 1. (1) Give an example of a ring that is not an integral domain. Give an example of an integral domain that is not a field. Can you find an example of a field that is not an integral domain?

(2) Consider the polynomial ring $\mathbb{Q}[x]$ and let $f(x) = -3 + 2x - 2x^2 + 2x^3 + x^4$ and $g(x) = x^2 + 1$. Does $g(x)$ divide $f(x)$? What is the greatest common divisor of $f(x)$ and $g(x)$?

Solution. (1) $6\mathbb{Z}$ is a ring but not an integral domain. $K[x]$ is an integral domain but not a field.

In a *field*, all nonzero elements are invertible with respect to multiplication. So, every field is an integral domain.

(2) Yes, $f(x) = (x^2 + 2x - 3)(x^2 + 1)$, making $\gcd(f, g) = g$ in this case.

Problem 2. Let $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with addition defined as $x \oplus y := \min(x, y)$ and multiplication $x \odot y := x + y$ for all $x, y \in \mathbb{R} \cup \{\infty\}$.

(a) Is $\mathbb{T}$ a commutative ring? If yes, then show that all axioms hold, if no, then explain which axiom fails.

(b) Calculate $3 \odot (5 \oplus 7)$, $(3 \oplus -3)^2$, and $(1 \oplus 8)^4$.

(c) Show that for any $x, y \in \mathbb{R} \cup \{\infty\}$, and any $k \in \mathbb{N}$, one has $(x \oplus y)^k = x^k \oplus y^k$.

Solution. For (a) note that $\min(x, \infty) = x$ for any $x \in \mathbb{T}$, which means that $0_\mathbb{T} = \infty$. The multiplicative unit is the "normal" additive unit, that is $1_\mathbb{T} = 0$. The multiplication $\odot$ is associative and commutative since the addition in $\mathbb{R} \cup \{\infty\}$ is associative and commutative. For the addition $\oplus$ write out:

$$a \oplus (b \oplus c) = a \oplus \min(b, c) = \min(a, \min(b, c)) = \min(a, b, c) = \min(\min(a, b), c) = (a \oplus b) \oplus c\,.$$

Commutativity is clear, because $\min(a, b) = \min(b, a)$. Thus $\oplus$ is associative, commutative and the neutral element is ∞. Distributivity comes from

$$a \odot (b \oplus c) = a + \min(b, c) = \min(a + b, a + c) = (a + b) \oplus (a + c) = (a \odot b) \oplus (a \odot c)$$

and the second equation follows trom commutativity of $\mathbb{T}$. However, not every element in $\mathbb{T}$ has an inverse with respect to $\oplus$: let $x \in \mathbb{R}$, then if x were invertible, there would be a $y \in \mathbb{T}$ such that $x \oplus y = \min(x, y) = \infty$. But $\min(x, y)$ is either y (if $y \leq x$) or x (if $x < y \leq \infty$) for any

$y \in \mathbb{T}$.
(b) $3 \odot (5 \oplus 7) = 3 + \min(5,7) = 8$, $(3 \oplus -3)^2 = (3 \oplus -3) \odot (3 \oplus -3) = \min(3,-3) + \min(3,-3) = -6$, and $(1 \oplus 8)^4 = (1 \oplus 8) \odot (1 \oplus 8) \odot (1 \oplus 8) \odot (1 \oplus 8) = 4\min(1,8) = 4$.
(c) First note that for any $x, y \in \mathbb{R} \cup \{\infty\}$, one has $k\min(x,y) = \min(kx,ky)$. Writing out $(x \oplus y)^k$ means $k\min(x,y) = \min(kx,ky)$. On the other hand, $x^k \oplus y^k = \min(kx,ky)$. So the two expressions are equal.

Problem 3. (a) Prove that if $\varphi : R \to S$ is a ring isomorphism then $\varphi^{-1} : S \to R$ is a ring homomorphism, and hence also an isomorphism.
(b) Let R be a ring and $I \subseteq R$ be an ideal and let $\varphi : R \to R/I$ be the canonical projection. Show that $\ker \varphi = I$ and φ is a ring homomorphism.

Solution. (a) Firstly if $\varphi(1_R) = 1_S$ then $\varphi^{-1}(1_S) = 1_R$. If now $s_1, s_2 \in S$ then there exist a unique pair $r_1, r_2 \in R$ with $\varphi(r_1) = s_1$ and $\varphi(r_2) = s_2$. Then

$$\begin{aligned} \varphi^{-1}(s_1 + s_2) &= \varphi^{-1}(\varphi(r_1) + \varphi(r_2)) \\ &= \varphi^{-1}(\varphi(r_1 + r_2)) \\ &= r_1 + r_2 \\ &= \varphi^{-1}(s_1) + \varphi^{-1}(s_2). \end{aligned}$$

Also

$$\begin{aligned} \varphi^{-1}(s_1 s_2) &= \varphi^{-1}(\varphi(r_1)\varphi(r_2)) \\ &= \varphi^{-1}(\varphi(r_1 r_2)) \\ &= r_1 r_2 \\ &= \varphi^{-1}(s_1)\varphi^{-1}(s_2). \end{aligned}$$

Therefore φ^{-1} is a homomorphism. Now since φ is a bijection so too is φ^{-1}, and therefore φ^{-1} is an isomorphism.
(b) Clearly we have $\varphi(1_R) = 1_R + I = 1_{R/I}$. If $r_1, r_2 \in R$ then

$$\begin{aligned} \varphi(r_1 + r_2) &= (r_1 + r_2) + I \\ &= (r_1 + I) + (r_2 + I) \\ &= \varphi(r_1) + \varphi(r_2), \end{aligned}$$

and

$$\begin{aligned}\varphi(r_1r_2) &= (r_1r_2) + I \\ &= (r_1 + I)(r_2 + I) \\ &= \varphi(r_1)\varphi(r_2).\end{aligned}$$

Finally, since $0_{R/I} = I$ and $r + I = I$ iff $r \in I$ we see that $r \in \ker\varphi \iff r + I = I \iff r \in I$. Therefore $\ker\varphi = I$.

Problem 4. Let I, J and K be ideals of a ring R. Show that
(a) $I \cap J$ and IJ are ideals
(b) $IJ \neq I \cap J$,
(c) $I(J + K) = IJ + IK$,

Solution. (a) Since I and J are ideal, both contain 0, and thus $0 \in I \cap J$, and $I \cap J \neq \varnothing$. Assume that $x, y \in I \cap J$ and $r \in R$. Since I and J are both ideals, it follows that $x \pm y$ and rx are in I and in J. Thus $x \pm y$ and rx are all in $I \cap J$. For the second assertion, note that IJ contains all finite sums of products of elements of I and J. Since $0 \in I$ and $0 \in J$, $0 = 0 \cdot 0 \in IJ$ and thus $IJ \neq \varnothing$. Let now $\sum_{i=1}^n x_iy_i$ and $\sum_{j=1}^m x'_jy'_j$ with $x_i, x'_j \in I$ and $y_j, y'_j \in J$. Then $\sum_{i=1}^n x_iy_i + \sum_{j=1}^m x'_jy'_j$ is clearly in IJ. If $r \in R$, then $r\left(\sum_{i=1}^n x_iy_i\right) = \sum_{i=1}^n (rx_i)y_i$ is also in IJ. Thus IJ is an ideal.
(b) We need only show one example where the above is not true. Therefore consider $I = 2\mathbb{Z}$, $J = 4\mathbb{Z} \subseteq \mathbb{Z}$. Then $IJ = 8\mathbb{Z}$ but $I \cap J = 4\mathbb{Z}$.
(c) Choose $x \in I(J + K)$, then x can be written as $\sum_{i=1}^n r_i(s_i + t_i)$ for some $n \in \mathbb{N}$, $r_i \in I$, $s_i \in J$ and $t_i \in K$. But then

$$\begin{aligned}x &= \sum_{i=1}^n (r_is_i + r_it_i) \\ &= \left(\sum_{i=1}^n r_is_i\right) + \left(\sum_{i=1}^n r_it_i\right) \\ &\in IJ + IK,\end{aligned}$$

so $I(J + K) \subseteq IJ + IK$. Conversely if $y \in IJ + IK$ then we can write $y = \sum_{i=1}^n r_is_i + \sum_{j=1}^m r'_jt_j$ for some $n, m \in \mathbb{N}$, $r_i, r'_j \in I$, $s_i \in J$ and $t_j \in K$. Now

$$\begin{aligned}y &= \sum_{\substack{i \\ r_i = r'_j}} r_i(s_i + t_j) + \sum_{\substack{i \\ r_i \neq r'_j \forall j}} r_i(s_i + 0) + \sum_{\substack{j \\ r'_j \neq r_i \forall i}} r'_j(0 + t_j) \\ &\in I(J + K),\end{aligned}$$

so $IJ + IK \subseteq I(J + K)$ and therefore $IJ + IK = I(J + K)$.

Problem 5. Let I, J and K be ideals of a ring R. Recall that $(I : J) = \{r \in R : rJ \subseteq I\}$. Show that
(a) $(I : J)$ is an ideal of R and $I \subseteq (I : J)$,
(b) $J \subseteq I$ implies that $(I : J) = R$,
(c) $IJ \subseteq K$ if and only if $I \subseteq (K : J)$.

Solution. (a) Clearly $0 \in (I : J)$ since $0J = \{0\} \subseteq I$. If $x_1, x_2 \in (I : J)$ then for all $y \in J$ we have $x_1y, x_2y \in I$, therefore $x_1y - x_2y = (x_1 - x_2)y \in I$. Hence $x_1 - x_2 \in (I : J)$. Finally if $x \in (I : J)$ and $r \in R$ then $rxJ \subseteq rI \subseteq I$, hence $rx \in (I : J)$.
If $x \in I$ and $y \in J$ then $xy \in I$, since $J \subseteq R$. Therefore $xJ \subseteq I$ and so $x \in (I : J)$ and $I \subseteq (I : J)$.
(b) If $J \subseteq I$ then $1J \subseteq I$, and $1 \in (I : J)$. But by part (a), $(I : J)$ is an ideal so $(I : J) = R$.
(c) Suppose first that $IJ \subseteq K$ and consider $x \in I$. Then for all $y \in J$ we have $xy \in IJ \subseteq K$, so $xJ \subseteq K$, i.e. $x \in (K : J)$ and $I \subseteq (K : J)$.
Conversely suppose $I \subseteq (K : J)$. Then $xy \in K$ for all $x \in I$ and $y \in J$. Hence all sums of the form $\sum_{i=1}^{n} x_iy_i$ with $n \in \mathbb{N}$, $x_i \in I$ and $y_i \in J$ are in K also, and hence $IJ \subseteq K$.

Problem 6. Let R be a commutative ring and let $I, J \subseteq R$ be ideals.
(a) Let $\sqrt{I} = \{r \in R : r^n \in I \text{ for some positive integer } n\}$. Show that $\sqrt{I}$ is an ideal that contains I. [Note: $\sqrt{I}$ is called the *radical of* I.]
(b) Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
(c) Let $R = k[x, y]$. Show that $\sqrt{(x^2, y^2)} = (x, y)$ and that $\sqrt{(x^2) \cap (y^2)} = (xy)$.

(a) We have to show that $\sqrt{I}$ is closed under addition and multiplication in R. Let $x \in \sqrt{I}$, then there exists an integer $n > 0$ such that $x^n \in I$. If $r \in R$ then $(rx)^n = r^nx^n$ is contained in I (since I is an ideal in R). This means that $rx \in \sqrt{I}$. If y is another element in $\sqrt{I}$, then there exists a $k > 0$ such that $y^k \in I$. Look at $(x + y)^{n+k}$. Use the binomial theorem:

$$(x+y)^{n+k} = \sum_{i=0}^{n+k} \binom{n+k}{i} x^i y^{n+k-i} = \underbrace{y^k \sum_{i=0}^{n-1} \binom{n+k}{i} y^{n+k-k-i} x^i}_{\in IR} + \underbrace{x^n \sum_{i=n}^{n+k} \binom{n+k}{i} y^{n+k-i} x^{i-n}}_{\in IR}$$

is contained in I. Thus $x + y \in \sqrt{I}$. Clearly $I \subseteq \sqrt{I}$, since for any $x \in I$, $x^1 \in \sqrt{I}$.
(b) First take $x \in \sqrt{I \cap J}$. This means that there exist $n > 0$ such that $x^n \in I$ and $x^n \in J$. This means that $x \in \sqrt{I}$ and $x \in \sqrt{J}$ and consequently in $\sqrt{I} \cap \sqrt{J}$. Now take x in the intersection of the two radicals. This means that there exist $k, l > 0$ such that $x^k \in I$ and $x^l \in J$. Then x^{k+l} is contained in both I and J. Thus $x \in \sqrt{I \cap J}$.
(c) It is easy to see that $\sqrt{(x^2, y^2)} \supseteq (x, y)$. If $f(x, y)$ is an element in $k[x, y]$ such that for some $n > 0$, $f^n = ax^2 + by^2$, then f cannot have a nonzero constant term. Thus f must be of the form

$f = cx + dy$ for some $c, d \in k[x, y]$. But this means that $f \in (x, y)$. For the second ideal use part (b): $\sqrt{(x^2) \cap (y^2)} = \sqrt{(x^2)} \cap \sqrt{(y^2)}$. Similar to the first ideal, one sees that $\sqrt{(x^2)} = (x)$ and $\sqrt{(y^2)} = (y)$, thus the ideal on the left hand side is $(x) \cap (y)$. Clearly $xy \in (x) \cap (y)$. For the other inclusion, if any $f(x, y) \in (x)$, then f is a multiple of x, i.e., $f(x, y) = xg(x, y)$ for some $g(x, y) \in k[x, y]$. But then $xg(x, y) \in (y)$ if and only if y is a factor of $xg(x, y)$, which means that y has to be a factor of $g(x, y)$. Thus $f(x, y) \in (xy)$ and we have shown the equality $(x) \cap (y) = (xy)$.