MATH3195/M5195 EXERCISE SHEET 3

DUE: MARCH 18, 2025

Problem 1. (a) Let $R = \mathbb{Q}[[x,y]]$ and let $J = \langle xy + y^3, x + x^2y, xy + 3y, x^4 - 5y^2 + x^2y\rangle$ be an ideal in R. Show that J is minimally generated by two elements in R.

(b) Let $R = K[t]$ and consider $M = K[t, t^{-1}]$ as R-module and let $I = tR$ be an ideal in R. Show that $M = IM$ but $M \neq 0$. Why does this example not contradict Nakayama's lemma?

(1) Let $R = K[x,y,z]$ and $I = \langle x^3, y^2, z\rangle$. Show that I is a primary ideal. Are $J = \langle x - y^3 - z^2\rangle$ and $L = \langle x + 1, y - 1, z + 1\rangle$ primary ideals? Explain!

(2) Show that $f = x^3 + x$ is integral over $K[x^2]$. Show that $K[x^2] \subseteq K[x]$ is an integral extension by finding an integral dependence relation for any $f \in K[x]$.

(3) Show that $\frac{x}{y}$ is integral over $K[x,y]/\langle x^2 - y^3\rangle$.

Problem 2. (a) Let $I = \langle x^2, y^2, z^2\rangle \cap \langle x + y\rangle \cap \langle x - y\rangle$ be an ideal in $R = \mathbb{R}[x,y,z]$. Is the given intersection of ideals a (minimal) primary decomposition of I? Explain!

(b) Let I be a *monomial ideal* in $\mathbb{Q}[x_1, \ldots, x_n]$, that is, I is generated by monomials. Show that if R is generated by pure powers of a subset of the variables, then it is a primary ideal. Further show that if $r = r_1r_2$ is a minimal generator of I, where r_1 and r_2 are relatively prime, then

$$I = (I + \langle r_1\rangle) \cap (I + \langle r_2\rangle)\ .$$

Remark: This yields an algorithm to compute primary decomposition of a monomial ideal!

Problem 3. (a) Let $R = \mathbb{R}[x,y]/\langle x^5 - y^3\rangle$. Show that $t = \frac{y}{x}$ and $u = \frac{x^2}{y}$ are integral over R. What are the R-module generators of $R[t]$ and $R[u]$?

(b) Let $f = x^3 + y^2$. Show that f is integral over $\mathbb{Q}[x^6, y^2]$.

(c) Let R be a unique factorisation domain, that is, R is an integral domain and every element in R can be written as a product of irreducible elements, unique up to order and multiplication with units. Show that every integral element of the form $\frac{x}{y}$, $x, y \in R$

is already contained in R. (*Remark*: this shows that R is *integrally closed* in its field of fractions).

Problem 4. Decompose $X := V((x^2y - xy^2)(x+y)) \subseteq \mathbb{A}^2_{\mathbb{R}}$ into irreducible components, that is, write X as a union of $V(f_i)$, where the f_i are irreducible polynomials. Same question for $X \subseteq \mathbb{A}^2_{\mathbb{F}_2}$, where $\mathbb{F}_2$ denotes the field with two elements.

Problem 5. (Will not be marked) Sketch the following affine algebraic sets for fun: use a computer algebra program for this!)

(a) $V(y^2 - x^5) \subset \mathbb{A}^2_{\mathbb{R}}$
(b) $V((x^2+y^2)^2 + 4x(x^2+y^2) - 4y^2) \subset \mathbb{A}^2_{\mathbb{R}}$
(c) $V(x^2+y^2-1) \subset \mathbb{A}^3_{\mathbb{R}}$,
(d) $V(x^3 + x^2z^2 - y^2) \subset \mathbb{A}^3_{\mathbb{R}}$
(e) $V(x^4y^2 - x^2y^4 - x^4z^2 + y^4z^2 + x^2z^4 - y^2z^4) \subset \mathbb{A}^3_{\mathbb{R}}$

Problem 6. Let $F = (x^2 - y^3)^2 - (z^2 - y^2)^3$ be a polynomial in $\mathbb{R}[x, y, z]$.

(1) Sketch $V(F) \subset \mathbb{A}^3_{\mathbb{R}}$.
(2) Let $J_F = \langle \partial_x(F), \partial_y(F), \partial_z(F) \rangle$ be the Jacobian ideal of F. Find $V(J_F)$ and sketch it.
(3) Is J_F radical?

Problem 7. The image of a non-constant complex polynomial map $f : \mathbb{C}^2 \to \mathbb{C}^3$ is a hypersurface. Let $f(s,t) = (s^3t^3, s^2, t^2)$.

(a) Find an irreducible polynomial map $F : \mathbb{C}^3 \to \mathbb{C}$ such that $\operatorname{Im}(f) \subset V(F)$. (Use coordinates (x, y, z).)
(b) Let again $J_F = \langle \partial_x(F), \partial_y(F), \partial_z(F) \rangle$ be the Jacobian ideal of F. Find a minimal primary decomposition of J_F and its associated primes. (Hint: Ensure J_F is simplified as much as possible and try to guess the primary components!)
(c) Hence show that J_F has an embedded prime and two isolated primes.