## MATH3195/M5195 EXERCISE SHEET 3

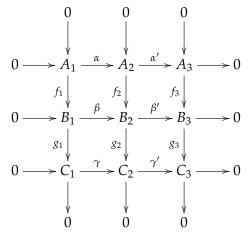
## DUE: MARCH 11, 2024

**Problem 1.** (a) Let  $R = \mathbb{Q}[[x, y]]$  and let  $J = \langle xy + y^3, x + x^2y, xy + 3y, x^4 - 5y^2 + x^2y \rangle$  be an ideal in *R*. Show that *J* is minimally generated by two elements in *R*.

(b) Let R = K[t] and consider  $M = K[t, t^{-1}]$  as *R*-module and let I = tR be an ideal in *R*. Show that M = IM but  $M \neq 0$ . Why does this example not contradict Nakayama's lemma?

Problem 2. Prove the first isomorphism theorem for modules.

**Problem 3.** Prove the  $3 \times 3$ -lemma: Let *R* be a ring. Assume that



is a commutative diagram of *R*-modules and all columns and the middle row is exact. Show that the top row is exact if and only if the bottom row is exact. **Problem 4.** (Localisation of a module) Let *R* be a ring and  $A \subset R$  be multiplicatively closed. Let *M* be an *R*-module. Assume we know that  $(m, a) \sim (n, b)$  if and only if mbc = nac for some  $c \in A$  defines an equivalence relation on  $M \times A$ . (Note: recall the definition of an equivalence relation.)

(a) Writing  $A^{-1}M$  for the set of equivalence classes of  $\sim$ , and  $\frac{m}{a}$  for the class containing (m, a), show that the operation

$$\frac{m}{a} + \frac{n}{b} = \frac{bm + an}{ab}$$

is well defined and hence that  $A^{-1}M$  is an abelian group.

(b) By defining an appropriate multiplication rule, show that  $A^{-1}M$  has the structure of an  $A^{-1}R$ -module.

**Problem 5.** Let *R* be a ring and  $A \subset R$  be multiplicatively closed.

- (a) Suppose that  $\phi : M \to N$  is a homomorphism of *R* modules. Show  $\phi$  induces an  $A^{-1}R$ -homomorphism  $A^{-1}M \to A^{-1}N$ .
- (b) Suppose  $0 \to L \to M \to N \to 0$  is an exact sequence of *R*-modules. Show that  $0 \to A^{-1}L \to A^{-1}M \to A^{-1}N \to 0$ , with the induced maps from (i), is an exact sequence of  $A^{-1}R$ -modules. (*Remark*: This means that localization is an exact functor from the category of *R*-modules to the category of  $A^{-1}R$ -modules.)