MATH3195/M5195 EXERCISE SHEET 2

DUE: MARCH 4, 2025

Problem 1. (a) Prove the first isomorphism theorem for modules.

(b) Show that $\mathbb{R}[x,y]/(x^3-y^2)$ is isomorphic to $\mathbb{R}[t^2,t^3]$. [Hint: First homomorphism theorem. First show that $f(x,y)=x^3-y^2$ is in the kernel of the map φ. In order to see that $(f(x,y))$ is the full kernel, you may use the fact, that the kernel of φ is generated by elements of the form $x^a y^b - x^{a'} y^{b'}$, where $a, a', b, b' \in \mathbb{N}$.]

(c) Is (x^3-y^2) a prime ideal in $\mathbb{R}[x,y]$? Explain!

Problem 2. (a) Consider $K[x,y,z]$ and order all monomials of degree less than or equal to 2 with respect to the following monomial orders: (i) $<_{lex}$, (ii) $<_{deglex}$, (iii) $<_\lambda$, where λ is a suitable linear form $\lambda : \mathbb{R}^3 \to \mathbb{R}$.

(b) Determine leading monomial and coefficient of the polynomial $f = x^4 + z^5 + x^3z + yz^4 + x^2y^2$ with respect to the momomial orders from (a).

Note: Do lots of examples using SAGE to understand orders in polynomials.

Problem 3. Let I be an ideal of R and A be a multiplicatively-closed subset of R. Show that:

(a) $A^{-1}I$ is an ideal of $A^{-1}R$;

(b) $\frac{x}{a} \in A^{-1}I$ if and only if there is some $b \in A$ with $xb \in I$;

(c) $A^{-1}I = A^{-1}R$ if and only if $I \cap A \neq \emptyset$;

(d) localisation commutes with quotients, that is

$$A^{-1}R/A^{-1}I \cong \overline{A}^{-1}(R/I),$$

where $\overline{A} = \{a + I : a \in A\}$.

Problem 4. (Localisation of a module) Let R be a ring and $A \subset R$ be multiplicatively closed. Let M be an R-module. Assume we know that $(m,a) \sim (n,b)$ if and only if $mbc = nac$ for some $c \in A$ defines an equivalence relation on $M \times A$. (Note: recall the definition of an equivalence relation.)

(a) Writing $A^{-1}M$ for the set of equivalence classes of $\sim$, and $\frac{m}{a}$ for the class containing (m, a), show that the operation

$$\frac{m}{a} + \frac{n}{b} = \frac{bm + an}{ab}$$

is well defined and hence that $A^{-1}M$ is an abelian group.

(b) By defining an appropriate multiplication rule, show that $A^{-1}M$ has the structure of an $A^{-1}R$-module.

Problem 5. (a) Let $0 \to A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \to 0$ and $0 \to B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \to 0$ be two short exact sequences of R-modules. Suppose that in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{\mu} & A & \xrightarrow{\varepsilon} & A'' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \alpha'} & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{\mu'} & B & \xrightarrow{\varepsilon'} & B'' & \longrightarrow & 0 \end{array}$$

α', α'' are isomorphisms. Then show that α is an isomorphism too.

(b) Prove the 3×3-lemma: Let R be a ring. Assume that

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\alpha} & A_2 & \xrightarrow{\alpha'} & A_3 & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\big\downarrow & & {\scriptstyle f_2}\big\downarrow & & {\scriptstyle f_3}\big\downarrow & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{\beta} & B_2 & \xrightarrow{\beta'} & B_3 & \longrightarrow & 0 \\ & & {\scriptstyle g_1}\big\downarrow & & {\scriptstyle g_2}\big\downarrow & & {\scriptstyle g_3}\big\downarrow & & \\ 0 & \longrightarrow & C_1 & \xrightarrow{\gamma} & C_2 & \xrightarrow{\gamma'} & C_3 & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

is a commutative diagram of R-modules and all columns and the middle row is exact. Show that the top row is exact if and only if the bottom row is exact.

(c) Give an example of two short exact sequences $0 \to A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \to 0$ and $0 \to B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \to 0$ with $A' \cong B'$ and $A'' \cong B''$ but where A is not isomorphic to B. Why does your example not contradict (a)?

Problem 6. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence of R-modules. Show that for any R-module A, the following

$$0 \to Hom_R(N, A) \xrightarrow{g^*} Hom_R(M, A) \xrightarrow{f^*} Hom_R(L, A)$$

is exact.