COMMUTATIVE RINGS AND ALGEBRAIC GEOMETRY

DUE: FEBRUARY 12, 2024

Exercise Sheet 1

Problem 1. Revision (have a look at the Rings and Polynomials MATH2027 notes, or equivalent course!): Make sure that you can answer the following:

(1) Give an example of a ring that is not an integral domain. Give an example of an integral domain that is not a field. Can you find an example of a field that is not an integral domain?
(2) Consider the polynomial ring $\mathbb{Q}[x]$ and let $f(x) = -3 + 2x - 2x^2 + 2x^3 + x^4$ and $g(x) = x^2 + 1$. Does $g(x)$ divide $f(x)$? What is the greatest common divisor of $f(x)$ and $g(x)$?

Problem 2. Let $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with addition defined as $x \oplus y := \min(x, y)$ and multiplication $x \odot y := x + y$ for all $x, y \in \mathbb{R} \cup \{\infty\}$.

(a) Is $\mathbb{T}$ a commutative ring? If yes, then show that all axioms hold, if no, then explain which axiom fails.
(b) Calculate $3 \odot (5 \oplus 7)$, $(3 \oplus -3)^2$, and $(1 \oplus 8)^4$.
(c) Show that for any $x, y \in \mathbb{R} \cup \{\infty\}$, and any $k \in \mathbb{N}$, one has $(x \oplus y)^k = x^k \oplus y^k$.

Problem 3. (a) Prove that if $\varphi : R \to S$ is a ring isomorphism then $\varphi^{-1} : S \to R$ is a ring homomorphism, and hence also an isomorphism.
(b) Let R be a ring and $I \subseteq R$ be an ideal and let $\varphi : R \to R/I$ be the canonical projection. Show that $\ker \varphi = I$ and φ is a ring homomorphism.

Problem 4. Let I, J and K be ideals of a ring R. Show that

(a) $I \cap J$ and IJ are ideals
(b) $IJ \neq I \cap J$,
(c) $I(J+K) = IJ + IK$,

Problem 5. Let I, J and K be ideals of a ring R. Recall that $(I : J) = \{r \in R : rJ \subset I\}$. Show that

(a) $(I : J)$ is an ideal of R and $I \subseteq (I : J)$,
(b) $J \subseteq I$ implies that $(I : J) = R$,
(c) $IJ \subseteq K$ if and only if $I \subseteq (K : J)$.

Problem 6. Let R be a commutative ring and let $I, J \subseteq R$ be ideals.

(a) Let $\sqrt{I} = \{r \in R : r^n \in I$ for some positive integer $n\}$. Show that $\sqrt{I}$ is an ideal that contains I. [Note: $\sqrt{I}$ is called the *radical of I*.]
(b) Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
(c) Let $R = k[x, y]$. Show that $\sqrt{(x^2, y^2)} = (x, y)$ and that $\sqrt{(x^2) \cap (y^2)} = (xy)$.