Problem 5. Let R be a ring and $A \subset R$ be a multiplicatively closed subset.

(a) Suppose that $\phi : M \to N$ is an R-mod. homomorphism. Show that ϕ induces an $A^{-1}R$ homo. $A^{-1}M \to A^{-1}N$.

Define $\psi : A^{-1}M \to A^{-1}N$

$$a^{-1}m \mapsto a^{-1}\phi(m)$$

- ψ is well-defined: If $a_1^{-1}m_1 = a_2^{-1}m_2$, then we have

$$\psi(a_1^{-1}m_1) = a_1^{-1}\phi(m_1) \overset{*}{=} \phi(a_1^{-1}m_1)$$

*since ϕ is an R-mod homomorphism

$$= \phi(a_2^{-1}m_2)$$
$$\overset{*}{=} a_2^{-1}\phi(m_2)$$
$$= \psi(a_2^{-1}m_2).$$

- (i) $\psi(a_1^{-1}m_1 + a_2^{-1}m_2) = \psi(a_1^{-1}a_2^{-1}(a_2m_1 + a_1m_2))$

$$= \psi((a_1a_2)^{-1}(a_2m_1 + a_1m_2))$$
$$= (a_1a_2)^{-1}\phi(a_2m_1 + a_1m_2)$$

*since ϕ is an R-mod homomorphism

$$\overset{*}{=} (a_1a_2)^{-1}(a_2\phi(m_1) + a_1\phi(m_2))$$
$$= a_1^{-1}\phi(m_1) + a_2^{-1}\phi(m_2)$$
$$= \psi(a_1^{-1}m_1) + \psi(a_2^{-1}m_2)$$

(ii) $r\psi(a^{-1}m) = r(a^{-1}\phi(m))$

$$= (ra^{-1})\phi(m)$$
$$= (a^{-1}r)\phi(m)$$
$$= a^{-1}(r\phi(m))$$
$$= a^{-1}\phi(rm)$$
$$= \psi(a^{-1}(rm))$$
$$= \psi(r(a^{-1}m))$$

we always use ϕ is an R-mod homo. and the module str. in R.

(b) Suppose $0 \to L \xrightarrow{\phi_1} M \xrightarrow{\phi_2} N \to 0$ is an exact sequence of R-modules. Show that

$0 \to A^{-1}L \xrightarrow{\psi_1} A^{-1}M \xrightarrow{\psi_2} A^{-1}N \to 0$ with induced maps from (i) is an exact sequence of $A^{-1}R$-mod.

- ψ_1 is injective:

$$\psi_1(a_1^{-1}l_1) = \psi_1(a_2^{-1}l_2)$$

$$\Rightarrow \quad a_1^{-1}\phi_1(l_1) = a_2^{-1}\phi_1(l_2)$$

$$\Rightarrow \quad \phi_1(a_1^{-1}l_1) = \phi_1(a_2^{-1}l_2)$$

$\Rightarrow \quad a_1^{-1} l_1 = a_2^{-1} l_2$ since ϕ_1 is injective

- ψ_2 is surjective

For every $a^{-1}n \in A^{-1}N$, there exists $a^{-1}m \in A^{-1}M$ such that

$$\psi_2(a^{-1}m) = a^{-1}n$$

since $\psi_2(a^{-1}m) = a^{-1}\phi_2(m) = a^{-1}n$ because ϕ_2 is surjective.

- $\text{Ker}\,\psi_2 = \text{Im}\,\psi_1$

(i) $\text{Ker}\,\psi_2 \subseteq \text{Im}\,\psi_1$

Take $a^{-1}m \in \text{Ker}\,\psi_2$

$\Rightarrow \quad \psi_2(a^{-1}m) = 0$

$\Rightarrow \quad a^{-1}\phi_2(m) = 0$

$\Rightarrow \quad \phi_2(a^{-1}m) = 0$

$\Rightarrow a^{-1}m \in \mathrm{Ker}\,\phi_2$

$\Rightarrow a^{-1}m \in \mathrm{Im}\,\phi_1$ since $\mathrm{Ker}\,\phi_2 = \mathrm{Im}\,\phi_1$

$\Rightarrow a^{-1}m = \phi_1(a_*^{-1}\ell)$ for some $a_*^{-1}\ell \in A^{-1}L$

$\Rightarrow a^{-1}m = a_*^{-1}\phi_1(\ell)$

$\Rightarrow a^{-1}m = \psi_1(a_*^{-1}\ell)$

$\Rightarrow a^{-1}m \in \mathrm{Im}\,\psi_1$

(ii) $\mathrm{Im}\,\psi_1 \subseteq \mathrm{Ker}\,\psi_2$

Take $a^{-1}m \in \mathrm{Im}\,\psi_1$

$\Rightarrow \psi_1(a_*^{-1}\ell) = a^{-1}m$ for some $a_*^{-1}\ell$

$\Rightarrow a_*^{-1}\phi_1(\ell) = a^{-1}m$

$\Rightarrow \phi_1(a_*^{-1}\ell) = a^{-1}m$

$\Rightarrow a^{-1}m \in \mathrm{Im}\,\phi_1$

$$\Rightarrow \quad a^{-1} m \in \operatorname{Ker} \phi_2 \quad \text{since } \operatorname{Ker}\phi_2 = \operatorname{Im}\phi_1$$

$$\Rightarrow \quad \phi_2(a^{-1} m) = 0$$

$$\Rightarrow \quad a^{-1} \phi_2(m) = 0$$

$$\Rightarrow \quad \psi_2(a^{-1} m) = 0$$

$$\Rightarrow \quad a^{-1} m \in \operatorname{Ker} \psi_2 .$$

Problem 3:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "\beta_1"] \arrow[d, "g_1"'] & B_2 \arrow[r, "\beta_2"] \arrow[d, "g_2"'] & B_3 \arrow[r] \arrow[d, "g_3"'] & 0 \\
0 \arrow[r] & C_1 \arrow[r, "\gamma_1"] \arrow[d] & C_2 \arrow[r, "\gamma_2"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Assume the first two rows are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d, "g_1"'] & B_2 \arrow[r] \arrow[d, "g_2"'] & B_3 \arrow[r] \arrow[d, "g_3"'] & 0 \\
 & \operatorname{coker} f_1 \arrow[r] \arrow[d] & \operatorname{coker} f_2 \arrow[r] \arrow[d] & \operatorname{coker} f_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\Rightarrow \quad 0 \to \operatorname{coker} f_1 \to \operatorname{coker} f_2 \to \operatorname{coker} f_3 \to 0$

$\operatorname{coker} f_i = C_i$ since columns are exact;

$$\operatorname{coker} f_i = B_i / \operatorname{Im} f_i \underset{\substack{\uparrow \\ \text{exactness}}}{\simeq} B_i / \ker g_i \underset{\substack{\uparrow \\ \text{isom. thm}}}{\simeq} \operatorname{Im} g_i \underset{\substack{\uparrow \\ \text{since } g \\ \text{is surjective}}}{=} C_i$$

Thus, we have $0 \to C_1 \to C_2 \to C_3 \to 0$ exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker g_1 \arrow[r] \arrow[d, "f_1"'] & \ker g_2 \arrow[r] \arrow[d, "f_2"'] & \ker g_3 \arrow[d, "f_3"'] \arrow[dddll, bend left] & \\
0 \arrow[r] & B_1 \arrow[r, "\beta_1"] \arrow[d, "g_1"'] & B_2 \arrow[r, "\beta_2"] \arrow[d, "g_2"'] & B_3 \arrow[r] \arrow[d, "g_3"'] & 0 \\
 & C_1 \arrow[r, "\gamma_1"] \arrow[d] & C_2 \arrow[r, "\gamma_2"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\Rightarrow \quad 0 \to \ker g_1 \to \ker g_2 \to \ker g_3 \to 0$$

$\ker g_i = A_i$ by the exactness of colums;

$\operatorname{Ker} g_i = \operatorname{Im} f_i = A_i$ since f_i's are injective

Thus, we have

$$0 \to A_1 \to A_2 \to A_3 \to 0 \quad \text{exact!}$$